Scale invariant quantum potential leading to globally self-trapped wave function in Madelung fluid

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Abstract. We show in spatially one dimensional Madelung fluid that a simple requirement on local stability of the maximum of quantum probability density will, if combined with the global scale invariance of quantum potential, lead to a class of quantum probability densities globally being self-trapped by their own self-generated quantum potentials, possessing only a finite-size spatial support. It turns out to belong to a class of the most probable wave function given its energy through the maximum entropy principle. We proceed to show that there is a limiting case in which the quantum probability density becomes the stationary-moving soliton-like solution of the Schrödinger equation.

PACS numbers: 03.65.Ge,03.65.Ca,03.65.Vf

1. Madelung fluid: global scale invariant quantum potential

Let us consider a spatially one dimensional Madelung fluid [1] for a single free particle with mass m. The state of the system is then determined by a pair of fields in position space q as: $\{\rho(q), v(q)\}$, where $\rho(q)$ is a normalized quantity called as quantum probability density and $v(q)$ is velocity field. The temporal evolution of both at time t is then assumed to satisfy the following coupled dynamical equation:

$$m\frac{dv}{dt} = -\partial_q U, \quad \partial_t\rho + \partial_q(\rho v) = 0. \tag{1}$$

Here, $U(q)$ is the so-called quantum potential generated by the quantum amplitude $R = \rho^{1/2}$ as

$$U(q) = -\frac{\hbar^2}{2m}\frac{\partial_q^2 R}{R}, \tag{2}$$

For the case of a spatially one dimensional fluid, the velocity field can always be written as the spatial gradient of a scalar function $S(q)$ as

$$v(q) = \partial_q S/m. \tag{3}$$

One can then use this new quantity to define a complex-valued wave function $\psi = R\exp(iS/\hbar)$ to show that the Madelung fluid dynamics given in Eq. (1) is equivalent to the Schrödinger equation for a single free particle as follows:

$$i\hbar\partial_t\psi(q;t) = -\frac{\hbar^2}{2m}\partial_q^2\psi(q;t). \tag{4}$$

Notice that while the equation on the right of Eq. (1) is but the conventional continuity equation which guarantees the conservation of probability flow, the left equation takes the form of Newtonian dynamical equation with a classically absence new term appears on the right hand side. In this regards, the term $F = -\partial_q U$ is called as quantum force. This fact suggests that the quantum force is the new quantity which is responsible for the nonclassical behaviors of Schrödinger equation. Thus, it is reasonable to pay serious attention to the the property of the quantum potential.

Let us mention an important properties of quantum potential that will play important roles in our discussion later. First is multiplying the quantum probability density with a constant will not change the profile of the quantum potential. Namely, the quantum potential is invariant under global rescaling of the quantum probability density, namely its own source, as:

$$U(c\rho(q)) = U(\rho(q)), \tag{5}$$

where c is constant. This shows that the quantum potential only cares about the form of the quantum probability density and is independent from the strength of the latter [2]. It is as if the quantum potential considers the quantum probability density as a code in telecommunication system in case of which only the profile of the sequence of the binary wave is important, the strength of the received wave itself is of no use. In this sense, the quantum potential is of informational nature. For an interesting and stimulating discussion concerning this matter see [2].

Among the consequences of the above invariant property is that, first rescaling the quantum probability density by the mass of the particle, $\tilde{\rho} = m\rho$ will not change the dynamical equation on the left of Eq. (1). On the other hand, the continuity equation on the right of Eq. (1) becomes

$$\partial_t \tilde{\rho} + \partial_q (v\tilde{\rho}) = 0, \tag{6}$$

which can now be read as the equation for the conservation of mass density, rather than the conservation of quantum probability density. The other consequence of the scale invariance property of quantum potential is that even at points where the strength of the quantum probability density is very low, the quantum potential that it generates at that point might be very high. In this paper, we shall be interested in a class of quantum probability densities with this specific property.

Next, let us mention another property of quantum potential $U(q)$ that the average of the quantum force, $F = -\partial_q U$, over the quantum probability is vanishing [3]

$$\int dq\, \partial_q U(q) \rho(q) = 0. \tag{7}$$

This can be proven easily by assuming $\rho(\pm\infty) = 0$. Imposing this into the dynamical equation of Eq. (1), one reproduces the Ehrenfest theorem [3]

$$m\frac{d\bar{v}}{dt} = 0, \tag{8}$$

where $\bar{v}$ is the average value of the velocity field defined as $\bar{v} \equiv \int dq\, v(q)\rho(q)$.

2. Local stability and globally self-trapping quantum potential

Let us now show that local geometrical restriction on the maximum point of the quantum probability density, if combined with the scale invariance of the quantum potential will determine the global geometrical property of the quantum potential, thus the quantum probability density as well. First, since the quantum probability density is vanishing at infinity, $\rho(\pm\infty) = 0$, non-negative and normalized, it must at least have one local maximum point. Let us denote this maximum point by $q = Q$. It thus satisfies

$$\partial_q \rho|_Q = 0, \;\; \partial_q^2 \rho|_Q < 0. \tag{9}$$

One first observes that at this point the quantum potential is positive definite

$$U(Q) = -\frac{\hbar^2}{4m}\Big(-\frac{1}{2}\Big(\frac{\partial_q \rho}{\rho}\Big)^2 + \frac{\partial_q^2 \rho}{\rho}\Big)\Big|_Q > 0. \tag{10}$$

Before proceeding, let us write a useful formula for later discussion

$$\frac{\partial_q^n \rho^s}{\rho^s}\Big|_Q = s\frac{\partial_q^n \rho}{\rho}\Big|_Q, \tag{11}$$

which can be shown easily by utilizing the left equation in (9) to be valid for any positive integer n.

Now, let us put a local restriction on a class of quantum probability densities $\rho(q)$ so that its maximum point stays at the minimum point of the quantum potential $U(q)$ which it generates through Eq. (2). One therefore imposes

$$\partial_q U|_Q = 0, \quad \partial_q^2 U|_Q \geq 0. \tag{12}$$

Dynamically we are thus looking for a class of quantum probability densities in which at least its maximum is temporally stable. Next, let us show that the restrictions given by Eqs. (12) will uniquely determine the form of $U(q)$ as a function of $\rho(q)$. First, from the global scale invariant property of the quantum potential, then the quantum force is also global scaling invariance; so that one has $\partial_q U(c\rho) = \partial_q U(\rho)$ for any real constant c. It is therefore reasonable to write the quantum force to take the following non-trivial form:

$$\partial_q U(\rho) = \frac{1}{\rho^s}(a_0 + a_1\partial_q + a_2\partial_q^2 + a_3\partial_q^3 + \dots)\rho^s, \tag{13}$$

where s and a_i, $i = 0, 1, 2, \dots$, are arbitrary real number. Evaluating at $q = Q$ and using the fact of Eq. (11) one has

$$\partial_q U(\rho(Q)) = a_0 + \frac{s}{\rho}(a_1\partial_q + a_2\partial_q^2 + a_3\partial_q^3 + \dots)\rho\Big|_Q. \tag{14}$$

The left equation in (12) imposes the right hand side of Eq. (14) to be vanishing. Keeping in mind Eqs. (9) and the fact that $\partial_q^n\rho|_Q$, for $n \geq 3$, are fluctuating between positive and negative value, $\partial_q U|_Q = 0$ can then be accomplished by imposing $a_0 = 0$, $a_j = 0$ for $j \geq 2$, and a_1 is arbitrary, yet non-vanishing. One therefore has

$$\partial_q U(\rho) = a_1\frac{\partial_q \rho^s}{\rho^s}. \tag{15}$$

Next, let us rewrite Eq. (15) as follows

$$\partial_q U(\rho) = a_1\partial_q \ln\rho^s = a_1 s\partial_q \ln\rho = a_1 s\frac{\partial_q\rho}{\rho} = a_s\frac{\partial_q\rho}{\rho}, \tag{16}$$

where we have denoted $a_s = a_1 s$. Now, taking spatial derivation on both sides of the above equation and using the left equation in Eq. (9), one gets

$$\partial_q^2 U\Big|_Q = a_s\frac{\partial_q^2\rho}{\rho}\Big|_Q - a_s\Big(\frac{\partial_q\rho}{\rho}\Big)^2\Big|_Q = a_s\frac{\partial_q^2\rho}{\rho}\Big|_Q. \tag{17}$$

Comparing this fact to the right inequality in (12) and keeping in mind the fact that $\partial_q^2\rho|_Q < 0$, one concludes that a_s must be non-positive. One can then verify that any quantum probability density that satisfies Eq. (16) satisfies all the requirements that we set at the beginning. Moreover, assuming $\rho(\pm\infty) = 0$, Ehrenfest theorem is automatically satisfied

$$\int dq\ \rho\partial_q U = a_s\int dq\ \partial_q\rho = 0. \tag{18}$$

To proceed, for simplicity of notation, let us rewrite Eq. (16) as follows

$$\partial_q U = -a\frac{\partial_q\rho}{\rho}, \ \ a \geq 0. \tag{19}$$

It can be readily integrated to obtain

$$\rho(q;a) = \frac{1}{Z(a)} \exp\Big(-\frac{1}{a}U(q;a)\Big), \tag{20}$$

where $Z(a) = \int dq\, \exp(-U/a)$ is a normalization constant independent of q. We shall show later that $U(q)$ can be interpreted as internal energy density. Bearing this in mind, then the quantum probability density given in Eq. (20) resembles in form with the Maxwell-Boltzmann-Gibbs (MBG) canonical distribution in equilibrium thermodynamics. It is thus suggestive to apply thermodynamics formalism to further study the property of quantum probability density given in Eq. (20) [5].

Next, let us recall that in quantum mechanics $\rho(q)$ gives the essential information on the position of the particle [2, 6, 7]. In the so-called pragmatical approach of quantum mechanics, $\rho(q)$ is given meaning as the probability density that the particle will be found at q if a measurement is performed. On the other hand, in the ontological approach, $\rho(q)$ is argued as the probability density that the particle is at q regardless of any measurement. It is thus reasonable to quantify the randomness encoded in $\rho(q)$. One obvious way is then to use the differential entropy or the so-called Shannon information entropy [8] over the quantum probability density given by

$$H[\rho] = -\int dq\, \rho(q) \ln \rho(q). \tag{21}$$

It gives the degree of localization of the wave function in position space.

One can then show that the canonical quantum probability density of the form given in Eq. (20) maximizes the Shannon entropy provided that the average quantum potential is given by [9]

$$\bar{U} = \int dq\, U(q)\rho(q). \tag{22}$$

Hence, the quantum probability density developed in the previous section satisfies the so-called maximum entropy principle [10]. It has been argued that the maximum entropy principle is the only method to infer from an incomplete information, which does not lead to logical inconsistency [11]. The self-trapped quantum probability density can then be seen as the most probable quantum probability density given its average quantum and kinetic energy [12].

Combined with the definition of quantum potential given in Eq. (2), Eq. (20) comprises a differential equation for $\rho(q)$ or $U(q)$ subjected to the condition that $\rho(q)$ must be normalized, $\int dq\rho(q) = 1$. In term of quantum potential, one has the following nonlinear differential equation

$$\partial_q^2 U - \frac{1}{2a}(\partial_q U)^2 - \frac{4ma}{\hbar^2}U = 0. \tag{23}$$

Figure 1 shows the solution of Eq. (23) with the boundary conditions: $\partial_q U(0) = 0$ and $U(0) = 1$ for $a = 1$. All numerical solutions in this paper are obtained by putting $m = \hbar = 1$. The quantum potential is shifted down so that its global minimum is vanishing. One can first see that the maximum point of the quantum probability density and the minimum point of the corresponding quantum potential coincide, thus satisfies

our requirement. Yet, what makes even interesting is that, though we only requires the quantum potential to trap the area of the quantum probability density around its maximum, it turns out that the resulting quantum potential is convex everywhere and becomes the global trapping potential for its own source: quantum probability density.

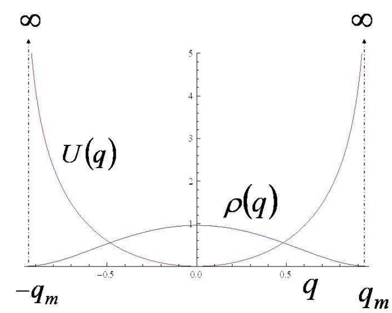

Figure 1. The profile of quantum probability density and its corresponding quantum potential which satisfies Eq. (23).

Next, one can also see that the solution plotted in Fig. 1 possesses blowing-up points at $q = \pm q_m$, namely $U(\pm q_m) = \infty$ [4]. Let us first prove that the blowing-up will certainly occur at finite point from the origin as along as $U(0) \equiv X$ is not vanishing. To do this, Let us define a new variable $u = \partial_q U$. The nonlinear differential equation of Eq. (23) then transforms into

$$\partial_q u = \frac{1}{2a} u^2 + \frac{4ma}{\hbar^2} U, \tag{24}$$

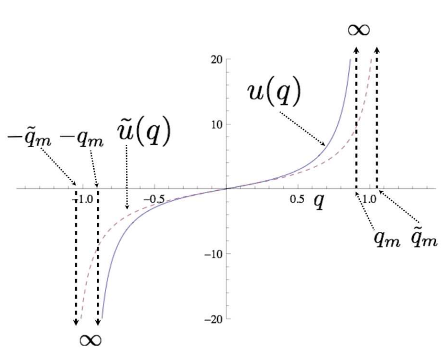

Figure 2. $\tilde{u}(q)$ and $u(q)$. See text for detail.

Moreover, the boundary condition translates into $u(0) = \partial_q U(0) = 0$. Let us now consider the following nonlinear differential equation

$$\partial_q \tilde{u} = \frac{1}{2a}\tilde{u}^2 + \frac{4ma}{\hbar^2}X, \tag{25}$$

where $X \equiv U(0)$ with $\tilde{u}(0) = 0$. Since $U(q) \geq U(0) = X$, then it is obvious that $|u(q)| \geq |\tilde{u}(q)|$.

On the other hand, one can solve the latter differential equation of Eq. (25) analytically to have

$$\tilde{u}(q) = d\tan(gq), \;\; d = \frac{2a}{\hbar}\sqrt{2m}, \;\; g = \frac{1}{\hbar}\sqrt{2mX}. \tag{26}$$

It is then clear that at $q = \pm\tilde{q}_m = \pm\pi/(2g)$, $\tilde{u}$ is blowing-up, namely $\tilde{u}(\pm\tilde{q}_m) = \pm\infty$. Recalling the fact that $|u(q)| \geq |\tilde{u}(q)|$, then $u(q)$ is also blowing-up at point $q = \pm q_m$, $u(\pm q_m) = \pm\infty$, where $q_m \leq \tilde{q}_m$. See Fig. 2. Putting this into the original nonlinear differential equation of Eq. (23), one concludes that $U(q)$ is also blowing up at $q = \pm q_m$, $U(\pm q_m) = \infty$. Notice that even though $u(-q_m) = -\infty$ is blowing-up to minus infinity, $U(-q_m) = \infty$ is obviously blowing up into positive infinity. Next, it is clear that the blowing-up is due to the existence of the nonlinear term on the right hand side of Eq. (23). Hence, finally one can safely say that for any non-vanishing $U(0) = X$, the corresponding quantum probability density possesses only a finite range of support, $q \in [-q_m, q_m]$. The case when $U(0) = 0$ will give the trivial solution $U(q) = 0$ for the whole space q so that $\rho(q)$ is unnormalizable. The above fact also confirms our assertion in Section I that the quantum potential might take large value even at points where the corresponding quantum probability density is very small.

In Fig. 3 we plot the blowing-up point $q = q_m$, namely half length of the spatial support of the quantum probability density against the value of the quantum potential at the global minimum: $U(0) = X$. One observes that q_m is decreasing as we increase X for fixed $a = 1$. This can be understood directly from Eq. (26). One can also confirm that the occurrence of blowing-up is the case only when $X \neq 0$, namely $\lim_{X\to 0} q_m = \infty$.

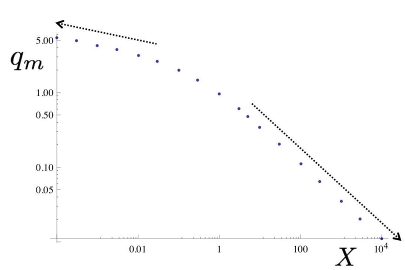

Figure 3. The half length of support q_m plotted against the variation of the global minimum $U(0) = X$.

3. Solitonic wave function

Now let us proceed to study the behavior of the quantum potential as one varies the non-negative parameter a. Figure 4 gives the variation of the blowing-up point, q_m, thus the size of the range of the support against the variation of the parameter a. This is obtained by solving the differential equation of Eq. (23) with fixed boundary conditions: $\partial_q U(0) = 0$ and $U(0) = 1$. One first observes that as a is increased, q_m decreases and eventually vanishing for infinite value of a. This shows that the quantum probability density is becoming narrower for larger a while kept normalized; and eventually collapsing onto Dirac delta function for infinite value of a.

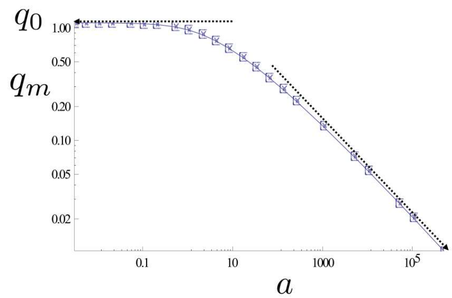

Figure 4. The half length of the support against the variation of a.

A very interesting phenomena is observed as one decreases the parameter a toward zero. One finds that the blowing up point q_m is increasing and eventually converging toward a finite value q_0 for $a = 0$,

$$\lim_{a\to 0} q_m(a) = q_0. \tag{27}$$

This suggests to us that the quantum potential and the corresponding quantum probability density are also converging toward certain functions for vanishing value of a:

$$\lim_{a\to 0} U(q;a) = U_0(q), \quad \lim_{a\to 0} \rho(q;a) = \rho_0(q). \tag{28}$$

Let us discuss this situation in more detail. In Figure 5 we plot the profile of the quantum probability density and the corresponding quantum potential for several small values of parameter a with fixed boundary conditions: $\partial_q U(0) = 0$ and $U(0) = 1$. One can then see that as a is decreased, the quantum potential is becoming flatterer inside the support before blowing-up at $q = \pm q_m(a)$. One might then guess that at the limit $a = 0$, the quantum potential is perfectly flat inside the support and is infinite at the blowing-up points, $q = \pm q_0$. Let us show that this guess is correct. To do this, let us denote the assumed constant value of the quantum potential inside the support as U_c. Recalling the definition of quantum potential given in Eq. (2), one has

$$-\frac{\hbar^2}{2m} R_0(q) = U_c R_0(q), \tag{29}$$

where $R_0 \equiv \rho_0^{1/2}$. This has to be subjected to the condition that $R_0(\pm q_0) = 0$.

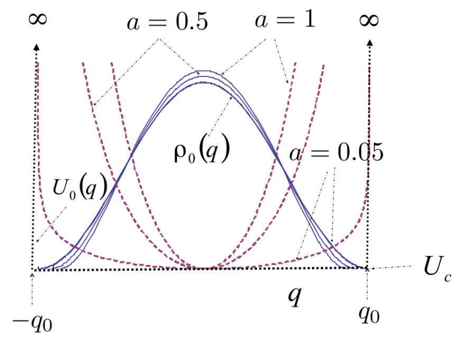

Figure 5. The profile of self-trapped quantum probability density and its corresponding quantum potential for several small values of parameter a. We also plot the case when $a = 0$ obtained analytically in Eq. (30).

Next, solving equation (29) one obtains

$$R_0(q) = A_0 \cos(k_0 q), \tag{30}$$

where A_0 is a normalization constant and k_0 is related to the quantum potential as

$$k_0 = \sqrt{2mU_c/\hbar^2}. \tag{31}$$

The boundary condition imposes $k_0 q_0 = \pi/2$. In Fig. 5, we plot the above obtained quantum probability density, $\rho_0(q)$. One can see that as a is decreasing toward zero, $\rho(q;a)$ obtained by solving the differential equation of (23) is indeed converging toward $\rho_0(q)$ given in equation (30). This confirms our guess that at $a = 0$ the quantum potential takes a form of flat box with infinite wall at $q = \pm q_0$.

Let us now take $\{\rho_0(q), v_0(q)\}$ as the initial state of the dynamics. Here $v_0(q;0) = v_c$ is a uniform velocity field with non-vanishing constant value only inside the support. Since at $a = 0$ the quantum potential is flat, then inside the support the quantum force is vanishing: $F = -\partial_q U = 0$. Inserting this into the dynamical equation of Eqs. (1), one has $dv/dt = 0$. Hence the velocity field at infinitesimal lapse of time, $t = \Delta t$, remains constant and uniform. This in turn will not change the initial probability density, but shift it in space by $\Delta q = v_c \Delta t$: $\rho(q;\Delta t) = \rho_0(q - v_c\Delta t; 0)$. Accordingly, the support is also shifted by the same amount. This will repeat in the next infinitesimal time lapse and so on and so forth so that at finite lapse of time t, both the initial velocity field and quantum probability density remains unchanged but are shifted by an amount $\Delta q = v_c t$. One thus has

$$\rho(q;t) = \rho_0(q - v_c t; 0) = A_0^2 \cos^2(k_0 q - \omega_0 t), \tag{32}$$

where we have put $\omega_0 = k_0 v_c$.

Before proceeding, let us give physical meaning to the average quantum potential. To do this, let us calculate the ordinary quantum mechanical energy given by $\langle E\rangle \equiv$

$\int_{-q_0}^{q_0} dq\, \psi^*(q)(-\hbar^2/2m)\partial_q^2\psi(q)$. Writing the wave function in polar form one has

$$\langle E\rangle = \int_{-q_0}^{q_0} dq \Big(-\frac{\hbar^2}{2m}R\partial_q^2 R + \frac{1}{2m}R^2(\partial_q S)^2 \\ -\frac{i\hbar}{m}R\partial_q R\partial_q S - \frac{i\hbar}{2m}R^2\partial_q^2 S\Big). \tag{33}$$

The first term on the right hand side is equal to the average quantum potential, $\bar{U} = \int dqU\rho$. Next, defining kinetic energy density as $K(q) = (m/2)v^2(q)$, the second term is equal to the kinetic energy $\bar{K} = \int dqK\rho$ of the Madelung fluid, which for our stationary state is given by $\bar{K} = (m/2)v_c^2$. Further, for a uniform velocity field, the last term is vanishing, $\partial_q^2 S = m\partial_q v_c = 0$. Again for a uniform velocity field, since $R(q)$ is an even function and $\partial_q R(q)$ is an odd function then the third term is also vanishing. Hence, in total, the quantum mechanical energy of a self-trapped wave function for $a = 0$ moving with a uniform velocity field can be decomposed as

$$\langle E\rangle = \bar{U}_0 + \bar{K}. \tag{34}$$

One can then conclude that $\bar{U}_0$ must essentially be interpreted as the rest energy of the single particle. Namely it is the energy of the particle when it is not moving so that $\bar{K} = 0$. Moreover, since inside the support the quantum potential is flat given by U_c, one has $\bar{U}_0 = \int_{-q_0}^{q_0} U_0(q)\rho(q) = U_c$. Recalling Eq. (31), one finally obtains

$$\langle E\rangle = \frac{\hbar^2 k_0^2}{2m} + \frac{1}{2}mv_c^2. \tag{35}$$

Let us now give the corresponding complex-valued stationary wave function $\psi(q)$. One thus needs to calculate the quantum phase S which can be obtained by integrating $\partial_q S = mv_c$ to give: $S(q;t) = mv_c q + \sigma(t)$, where $\sigma(t)$ depends only on time. One therefore has $\psi_{st}(q;t) = A_0\cos(k_0(q - v_c t))\exp\Big((i/\hbar)(mv_c q + \sigma(t))\Big)$, where $q \in \mathcal{M}_t \equiv [v_c t - q_0, v_c t + q_0]$. Inserting this into the Schrödinger equation of Eq. (4) and using Eq. (35) one has $\sigma(t) = -\langle E\rangle t$ modulo to some constant. Putting all these back, one finally obtains the following solution:

$$\psi_{st}(q;t) = A_0\cos(k_0(q - v_c t))\exp\Big(\frac{i}{\hbar}(mv_c q - \langle E\rangle t)\Big), \tag{36}$$

where $q \in \mathcal{M}_t$.

Equation (36) is of soliton type. This suggests a direct association of the wave function to a particle by considering the wave function as a *physical field*. In this regard, the continuity equation of mass density of Eq. (6) is becoming relevant. On the other hand, we have also shown in the previous section that the localized-stationary-traveling solution belongs to a class of wave function which maximizes Shannon entropy, which suggests that it is a *probabilistic wave field*. Hence, one arrives at one of the old problem of quantum mechanics concerning the physical status of the wave function.

4. Conclusion and Interpretation

By exploiting the scaling invariant property of the quantum potential, we show that the requirement of local stability on the maximum of quantum probability density leads

us to a class of quantum probability densities which is globally trapped by its own quantum potential with finite-size spatial support. It turns out that they belong to a class of wave function which maximizes Shannon entropy given the average quantum potential. Further, we show that for a single free particle quantum system, there is an asymptotic limit in which the self-trapped wave function is traveling while keeping its form unchanged. This fact thus suggests to us to associate the localized-traveling quantum probability density as a real particle. In contrast to this, in conventional formalism of quantum mechanics, one usually choose a plane wave to represent a free single particle.

In our formalism of a single particle as a localized and self-trapped wave function, we showed that a particle possesses an internal energy which is absence if one use a plane wave instead. It is equal to the quantum potential. On the other hand, we showed in the beginning of the paper that the quantum potential is invariant under the global rescaling of its own source, namely the quantum probability density. It depends only on the form of the latter. It is a surprising fact, since, usually energy is an extensive quantity with respect to its source. Hence, the internal energy is of different nature from the ordinary one. Since it only records the profile of its own source, one might conclude that its nature is informational, rather than material.

An interesting point is left unexplored. On one hand, we showed that the canonical form of quantum probability density given in Eq. (20) is the consequence of the scale invariant property of the quantum potential. On the other hand, we also showed that it maximizes Shannon information entropy given its average quantum potential. One may then expect that these two facts are related in a nontrivial way.

Acknowledgments

References

[1] Madelung E 1926 *Zeits. F. Phys.* **40** 332
[2] Bohm D and Hiley B J 1993 *The Undivided Universe: An ontological interpretation of quantum theory* (London: Routledge)
[3] Holland P R 1993 *The Quantum Theory of Motion* (UK: Cambridge University Press)
[4] Alinhac S 1995 *Blowup for Nonlinear Hyperbolic Equations* (Basel: Birkhäuser)
[5] Budiyono A and Umeno K submitted for publication
[6] Isham C J 1995 *Lectures On Quantum Theory: Mathematical and Structural Foundation* (London: Imperial College Press)
[7] Bell J. S 1987 *Speakable and Unspeakable in Quantum Mechanics* (UK: Cambridge University Press)
[8] Shannon C E 1948 *Bell Systems Technical Journal* **27** 379
[9] Mackey M C 1989 *Reviews of Modern Physics* **61** 98
[10] Jaynes E T 1949 *Physical Review* **106** 620
[11] Shore J E and Johnson R W 1980 *IEEE Transaction on Information Theory* **IT-26** 26
[12] Budiyono A and Umeno K 2009 *Phys. Rev. A* **79** 210421